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A. V. Gorshkov, and E. Yu. Prosviryakov

Convective Flow in the Solid Rotation of a Viscous Incompressible Fluid

A. V. Gorshkov[1, 2 a)] and E. Yu. Prosviryakov[1, 2, 3 b)]

[1]*Institute of Engineering Science, Ural Branch of the Russian Academy of Science, 34 Komsomolskaya St., Ekaterinburg, 620049, Russia.*
[2]*B. N. Yeltsin Ural Federal University, 19 Mira St., Ekaterinburg, 620002, Russia.*
[3]*Kazan National Research Technical University named after A. N. Tupolev (KNRTU-KAI), 10 K. Marksa St., Kazan, Republic of Tatarstan, 420111, Russia.*

[a)]Corresponding author: alex55gor@mail.ru
[b)]evgen_pros@mail.ru

Abstract. The analytical solution of the Ekman convective stationary flow of a viscous incompressible fluid in an infinite layer is obtained. A solution to an overdetermined system of the Oberbeck-Boussinesq equations is considered. It is shown that the structure of the solution allows one to preserve the advective derivative in the heat equation; this makes it possible to model the delamination of the temperature and pressure fields and to describe backflow in the ocean.

INTRODUCTION

The Ekman stationary large-scale flow of a non-uniformly heated viscous incompressible fluid is described by a system of Oberbeck-Boussinesq equations [1-5], which is written in dimensionless form as

$$\begin{gathered}
\mathrm{Gr}\left(V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}\right)-\frac{1}{\mathrm{Ek}}V_y=-\frac{\partial P}{\partial x}+\Delta V_x,\\
\mathrm{Gr}\left(V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}\right)+\frac{1}{\mathrm{Ek}}V_x=-\frac{\partial P}{\partial y}+\Delta V_y,\\
\frac{\partial P}{\partial z}=T,\quad V_x\frac{\partial T}{\partial x}+V_y\frac{\partial T}{\partial y}=\frac{1}{\mathrm{Ra}}\Delta T,\quad \frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}=0.
\end{gathered}\tag{1}$$

Here, V_x, V_y are the components of the dimensionless fluid velocity vector, $V = g\beta\vartheta L^2/\nu$ is the characteristic scale of velocity; the dimensionless horizontal coordinates x and y are defined by the characteristic scale length L, and the transverse coordinate z is determined by the thickness h of the fluid layer; $\delta = h/L$ is the ratio of the scale length; $\mathrm{Gr} = g\beta\vartheta L^3/\nu^2$ is the Grashof number; β is the fluid volume expansion coefficient; g is free fall acceleration; ϑ is the difference between the maximum and minimum temperatures, ν is the kinematic (molecular) viscosity coefficient; Ra is the Rayleigh number; Pr is the Prandtl number; χ is thermal diffusivity; Ek is the Ekman number; $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{\delta^2}\frac{\partial^2}{\partial z^2}$ is the Laplace operator.

On the lower boundary of the fluid layer ($z=0$) the adhesion condition is met and heating is set,

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$$V_x = 0, \quad V_y = 0, \quad T = xT_g \cos\psi + yT_g \sin\psi. \tag{2}$$

On the upper boundary ($z = 1$), which is non-deformable and isothermal, the following conditions are set:

$$V_x = W\cos\varphi, \quad V_y = W\sin\varphi, \quad T = 0, \quad P = 0. \tag{3}$$

Here, W is the magnitude of the dimensionless velocity on the fluid surface. The vector of the velocity of the upper boundary of the fluid layer makes an angle φ with the Ox axis.

EXACT SOLUTION

The stationary solution of system (1) has the following form [5-11]:

$$V_x = U(z), \quad V_y = V(z), \quad T = T_0(z) + xT_1(z) + yT_2(z), \quad P = P_0(z) + xP_1(z) + yP_2(z). \tag{4}$$

The representation of hydrodynamic fields (4) enables one to find a nontrivial solution to the overdetermined Oberbeck-Boussinesq equation system (1), since the velocity field (4) identically satisfies the incompressibility equation (1).

A system of ordinary differential equations is obtained after the substitution of formulas (4) into Eq. (1),

$$T_1'' = 0, \quad T_2'' = 0, \quad P_1' = T_1, \quad P_2' = T_2, \tag{5}$$

$$U'' + 2k^2\delta^2 V = P_1, \quad V'' - 2k^2\delta^2 U = P_2, \tag{6}$$

$$T_0'' = \mathrm{Ra}\,\delta^2 \left(T_1 U + T_2 V\right), \quad P_0' = T_0. \tag{7}$$

The system of ordinary differential equations is given in the order of integration. Here, $2k^2 = 1/\mathrm{Ek}$, the prime denoting differentiation with respect to the z coordinate.

The boundary conditions represented by Eqs. (2) and (3) within the class represented by Eq. (4) can be written as follows:

$$U = 0\,, \; V = 0\,, \; T_0 = 0\,, \; T_1 = T_g \cos\psi\,, \; T_2 = T_g \sin\psi\,, \text{ when } z = 0\,; \tag{8}$$

$$U = W\cos\varphi, \;\; V = W\sin\varphi, \;\; T_1 = 0, \;\; T_2 = 0, \;\; T_0 = 0, \;\; P_0 = 0, \;\; P_1 = 0, \;\; P_2 = 0, \text{ when } z = 1. \tag{9}$$

According to the boundary conditions (9), (10) the solution of equations (5) can be presented as

$$T_1 = T_g \cos\psi(1 - z)\,, \; T_2 = T_g \sin\psi(1 - z)\,; \; P_1 = T_g \cos\psi\left(z - \left(z^2 + 1\right)/2\right), \; P_2 = T_g \sin\psi\left(z - \left(z^2 + 1\right)/2\right). \tag{10}$$

The solution of the system of equations (7), which satisfies the boundary conditions (9), (10), can be presented in the form

$$U(k,z)=\frac{e^{-\delta kz}}{1+e^{4\delta k}-2e^{2\delta k}\cos(2\delta k)}\Big\{W\,e^{\delta k}\left(u_{W1}(k,z)\cos\varphi+u_{W2}(k,z)\sin\varphi\right)+$$
$$+T_g\left[\left(\frac{\mathrm{Ek}}{\delta^2}\,u_{T1}(k,z)+\left(\frac{\mathrm{Ek}}{\delta^2}\right)^2 u_{T2}(k,z)\right)\cos\psi+\left(\frac{\mathrm{Ek}}{\delta^2}\,u_{T3}(k,z)+\left(\frac{\mathrm{Ek}}{\delta^2}\right)^2 u_{T4}(k,z)\right)\sin\psi\right]\Big\},$$
$$V(k,z)=\frac{e^{-\delta kz}}{1+e^{4\delta k}-2e^{2\delta k}\cos(2\delta k)}\Big\{W\,e^{\delta k}\left(-u_{W2}(k,z)\cos\varphi+u_{W1}(k,z)\sin\varphi\right)+$$
$$+T_g\left[-\left(\frac{\mathrm{Ek}}{\delta^2}\,u_{T3}(k,z)+\left(\frac{\mathrm{Ek}}{\delta^2}\right)^2 u_{T4}(k,z)\right)\cos\psi+\left(\frac{\mathrm{Ek}}{\delta^2}\,u_{T1}(k,z)+\left(\frac{\mathrm{Ek}}{\delta^2}\right)^2 u_{T2}(k,z)\right)\sin\psi\right]\Big\}. \qquad (11)$$

The notation for the solution (12) is as follows:

$$u_{W1}(k,z)=-\left(e^{2\delta k}+e^{2\delta kz}\right)\cos\left(\delta k\,(1+z)\right)+\left(1+e^{2\delta k(1+z)}\right)\cos\left(\delta k\,(1-z)\right),$$
$$u_{W2}(k,z)=-\left(e^{2\delta k}-e^{2\delta kz}\right)\sin\left(\delta k\,(1+z)\right)+\left(-1+e^{2\delta k(1+z)}\right)\sin\left(\delta k\,(1-z)\right),$$
$$u_{T1}=-\frac{e^{\delta k}}{2}\left(e^{2\delta k}\left(e^{2\delta k(z-1)}-1\right)\sin\left(\delta k\,(1+z)\right)+\left(e^{2\delta k(1+z)}-1\right)\sin\left(\delta k\,(1-z)\right)\right),$$
$$u_{T2}=\left(\left(e^{2\delta k}+e^{2\delta kz}\right)\cos(\delta kz)-e^{\delta kz}\left(1+e^{2\delta k}+2e^{\delta k}\cos(\delta k)\right)+\right.$$
$$\left.+e^{\delta k}\left(1+e^{2\delta kz}\right)\cos\left(\delta k\,(1-z)\right)\right)\times\left(1+e^{2\delta k}-e^{\delta k}\cos(\delta k)\right),$$
$$u_{T3}=\left(e^{\delta kz}\left(1+e^{4\delta k}-2e^{2\delta k}\cos(2\delta k)\right)z(z-2)-e^{\delta k}\left(e^{2\delta k}+e^{2\delta kz}\right)\cos\left(\delta k\,(1+z)\right)+\right.$$
$$\left.+e^{\delta k}\left(1+e^{2\delta k(1+z)}\right)\cos\left(\delta k\,(1-z)\right)\right),$$
$$u_{T4}=\left(e^{2\delta k}\left(1-e^{2\delta k(z-1)}\right)\sin(\delta kz)+e^{\delta k}\left(e^{2\delta kz}-1\right)\sin\left(\delta k\,(1-z)\right)\right)\times\left(1+e^{2\delta k}-2e^{\delta k}\cos(\delta k)\right).$$

The solution of equation (7) is obtained by double integration of the given solution (11) of the system (5) with the conditions (8), (9), and it has the form

$$T_0(k,z)=\frac{\mathrm{Ek}\,\mathrm{Ra}\,T_g\,e^{-\delta kz}}{\delta k\left(1+e^{4\delta k}-e^{2\delta k}\cos(2\delta k)\right)}\left\{T_g\left(\frac{\mathrm{Ek}}{\delta^2}T_{02}(k,z)+\left(\frac{\mathrm{Ek}}{\delta^2}\right)^2 T_{03}(k,z)\right)+W\,T_{0W}\left(k,z,\varphi-\psi\right)\right\}. \qquad (12)$$

Here, the functions $T_{02}(\delta k,z)$, $T_{03}(\delta k,z)$ and $T_{0W}(\delta k,z,\varphi-\psi)$ are quasi-polynomials of the variable z.

FLUID FLOW ANALYSIS

The isothermal solution has the form

$$U=\frac{W\,e^{\delta k(1-z)}\left(u_{W1}(k,z)\cos\varphi+u_{W2}(k,z)\sin\varphi\right)}{1+e^{4\delta k}-2e^{2\delta k}\cos(2\delta k)},\quad V=\frac{W\,e^{\delta k(1-z)}\left(u_{W1}(k,z)\sin\varphi-u_{W2}(k,z)\cos\varphi\right)}{1+e^{4\delta k}-2e^{2\delta k}\cos(2\delta k)}.$$

This exact solution was described in [1, 5]. Let us now turn to the analysis of the rotating fluid (11). The examination of Eq. (11) shows that, when $k>4$, there is one stagnation point in the fluid (Fig. 1). When $k>5.5$, the velocity field can be split into three zones (Fig. 1), this being equivalent to the existence of two stagnation points for velocities. The flow is spiral in all the cases.

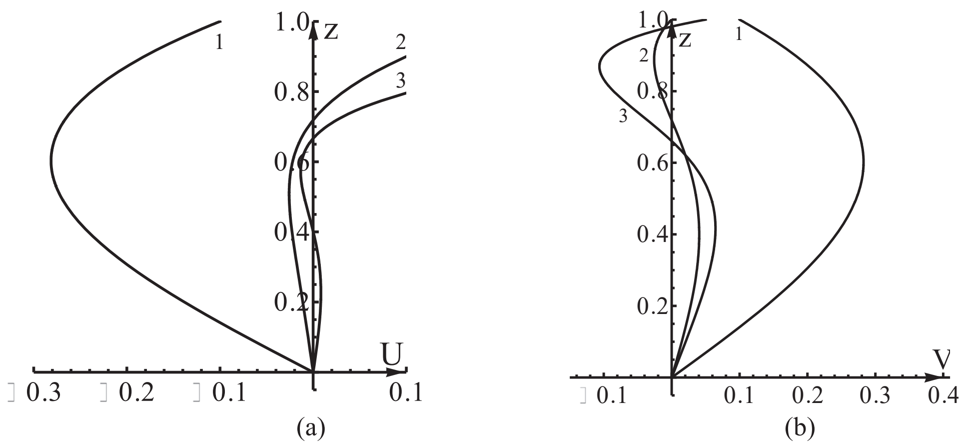

FIGURE 1. The velocity components *U* and *V* as functions of the following parameters:
1 – flow without stagnation points, 2 – flow with one stagnation point and 3 – flow with two stagnation points

The flow parameters are given in table 1.

TABLE 1 The values of the flow parameters

	k	$T_g\cos\psi$	$T_g\sin\psi$	W_x	W_y
1	0.1	5.0	-5.0	-0.1	0.1
2	4.1	5.0	0.0	0.2	0.0
3	5.5	11.0	-4.5	0.6	0.05

The temperature field, whose topological structure is determined by the function T_0, is characterized by nonmonotonic profiles (Fig. 2). The temperature distribution is determined by two cases. Firstly, the temperature takes values of the same sign. In other words, the temperature decreases monotonically through the layer thickness. Secondly, the separation temperature is possible by analogy with the velocity field. In this case, a thermocline is observed in the fluid. The results of the pressure analysis are similar to those obtained for temperature; therefore, they are not reported here.

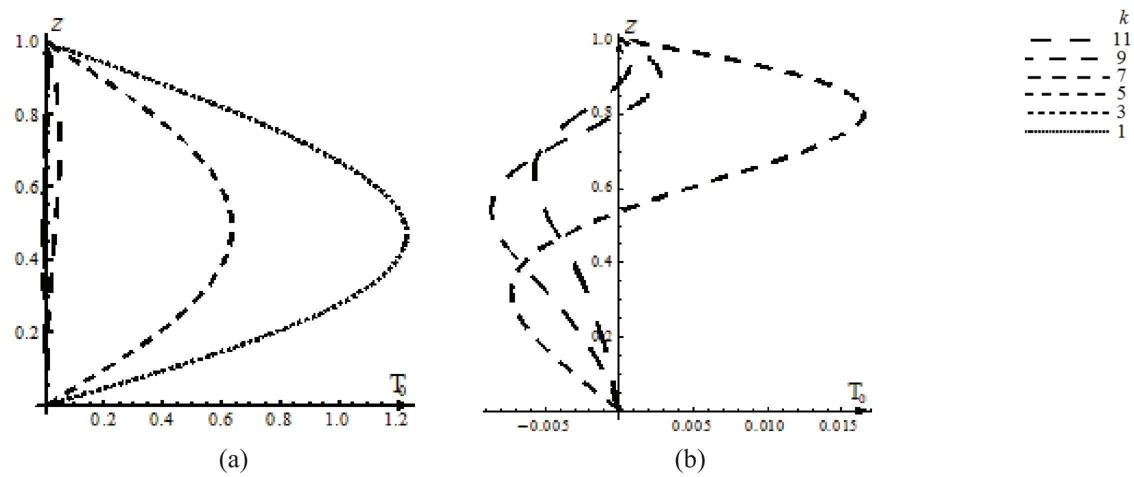

FIGURE 2. The *z*- and *k*- dependence of temperature for $T_g = 5$, $W = 0.2$, $\varphi = 3\pi/4$, $\psi = 0$, $\delta^2 Ra = 15$

CONCLUSION

Exact (analytic) solutions describing convection in the infinitely extended layer of a viscous incompressible fluid with consideration of the Coriolis force have been obtained. The obtained expressions for hydrodynamic fields have been compared with those for the isothermal fluid flow in the equatorial zone and in the Coriolis force field. The occurrence of backflow and the stratification of the temperature and pressure fields have been demonstrated. $\varphi = 3\pi / 4$.

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